Power Density at distance d from a radiator: $P = \frac{P_t}{4\pi d^2}$; Far-field (far from transmitter) power density: $P = \frac{P_t G_t}{4\pi d^2}$ $c = f \cdot \lambda, \Delta\phi = \frac{2\pi\Delta d}{\lambda}$

Ideal Dish Antenna: $P_r(d) = P = \frac{P_t G_t A_d}{4\pi d^2}$; for practical antennas, A_e replaces area; $A_e = \frac{G_r \lambda^2}{4\pi}$

Received Power at distance d (Friis Free Space Equation): $P_r(d) = \frac{P_t G_t G_r \lambda^2}{(4\pi d)^2}$

Decibel Notation: $G_{dB} = 10\log_{10} G$; $G = 10^{\frac{G_{dB}}{10}}$;

$P_{dBm} = 10\log_{10} P_{mW}$; $P_{dBW} = 10\log_{10} P_W$; $P_{dBm} = 10\log_{10}(1000P_{mW}) = 30 + P_{dBW}$

Exponential Path Loss: $PL_{dB} = 20\log_{10} d + 20\log_{10} f - 147.6\ dB$; with exponential n: $PL_{dB} = 10n\log_{10} d + 20\log_{10} f - 147.6\ dB$

$$[P_r(d)]_{dBm} = [P_r(d_0)]_{dBm} - 20\log_{10}\frac{d}{d_0}; \text{ with exponential } n: [P_r(d)]_{dBm} = [P_r(d_0)]_{dBm} - 10n\log_{10}\frac{d}{d_0}$$

SIGNAL ENERGY: $E_g = \int_{-\infty}^{\infty}|g(t)|^2 dt$; **SIGNAL POWER:** $P_g = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|g(t)|^2 dt$; RMS = $\sqrt{P_g}$

Inner Product: $\langle g, x\rangle = \|g\| \cdot \|x\| \cdot cos(\theta)$; Norm of $x = \langle x, x\rangle = \|x\|^2$, $c\|x\| = \|g\| cos(\theta) \to c = \frac{\langle g,x\rangle}{\langle x,x\rangle}$

Error: inner product of the function with predicted function $e(t) = \begin{cases} g(t) - cx(t), t_1 \le t \le t_2 \\ 0, otherwise \end{cases}$; $c = \frac{\int_{t_1}^{t_2} g(t)x(t)}{\int_{t_1}^{t_2} x^2(t)} dt$ or $c = \frac{1}{\sqrt{E_x}}\int_{t_1}^{t_2} g(t)x^*(t)\,dt$ (for complex);

if $\langle g, x\rangle = \sum_{i=1}^{\infty} g_i x_i$, then $\langle g(t), x(t)\rangle = \int_{t_1}^{t_2} g(t)x(t)\,dt$; $g(t) \approx \sum_i c_i \cdot f_i(t)$ $C_i = \frac{\langle g, fi\rangle}{\|fi\|}$

Cross-correlation: similarity of $g(t)$ and $z(t)$ left-shifted by τ: $\psi_{zg}(\tau) \equiv \int_{-\infty}^{\infty} z(t)g^*(t-\tau)\,dt = \int_{-\infty}^{\infty} z(t+\tau)g^*(t)dt =$. Autocorrelation: with itself: $\psi_g(\tau) = \int_{-\infty}^{\infty} g(t)\cdot g(t+\tau)\,dt$

Fourier synthesis: $S(t) = \sum_{n=-\infty}^{\infty} a_n e^{\frac{j2\pi nt}{T}}$; Generalized Fourier Series: $\sum_{n=1}^{\infty} c_n x_n(t)$; Parseval's Theorem: $E_g = \sum_n c_n^2 E_n$

Trig. Fourier Series: $g(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$ where $a_0 = \frac{1}{T_0}\int_{t_1}^{t_1+T} g(t)dt$, $a_n = \frac{2}{T_0}\int_{t_1}^{t_1+T} g(t)\cos n2\pi f_0 t\,dt$, $b_n = \frac{2}{T_0}\int_{t_1}^{t_1+T} g(t)\sin n2\pi f_0 t\,dt$; ** $\int_{T_0} \cos n\omega_0 t \cos m\omega_0 t = \begin{cases} 0, n \ne m \\ \frac{T_0}{2}, n = m \ne 0 \end{cases}$; $\int_{T_0} \sin n\omega_0 t \sin m\omega_0 t = \begin{cases} 0, n \ne m \\ \frac{T_0}{2}, n = m \ne 0 \end{cases}$; $\int_{T_0} \sin n\omega_0 t \cos m\omega_0 t = 0$

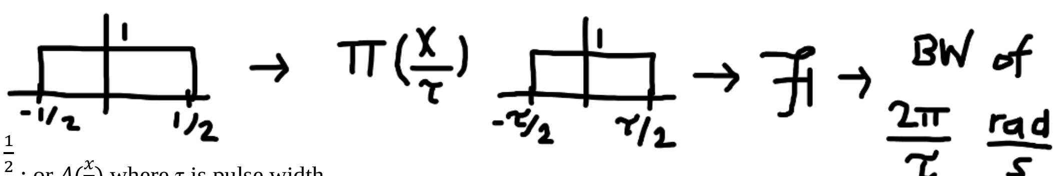
Compact Fourier Series: $a_n \cos n2\pi f_0 t + b_n \sin n2\pi f_0 t = C_n \cos(n2\pi f_0 t + \theta_n)$, where $a_0 = C_0$, $C_n = \sqrt{a_n{}^2 + b_n{}^2}$, $\theta_n = \tan^{-1}(\frac{-b_n}{a_n})$; $P_g = C_0{}^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n{}^2$

Exponential Fourier Series: $g(t) = \sum_{n=-\infty}^{\infty} e^{-jn2\pi f_0 t}$; $D_n = \frac{1}{T_0}\int_{T_0} g(t)e^{-jn2\pi f_0 t}$; $P_g = \sum_{n=-\infty}^{\infty}|D_n|^2 \to D_0{}^2 + 2\sum_{n=1}^{\infty}|D_n|^2$ (if $g(t)$ is real)

Parseval's Theorem: $\int_{-\infty}^{\infty}|g(t)|^2\,dt = \int_{-\infty}^{\infty}|g(f)|^2\,dt$

Existence of Fourier Transform: $\int_{-\infty}^{\infty}|g(t)|\,dt < \infty$

Rectangular Function: $\Pi(x) = \begin{cases} 1\ if\ |x| \le \frac{1}{2} \\ 0\ if\ |x| > \frac{1}{2} \end{cases}$;



Triangular Function: $\Delta(x) = \begin{cases} 1 - 2|x|\ if\ |x| < \frac{1}{2} \\ 0\ if\ |x| > \frac{1}{2} \end{cases}$; or $\Delta(\frac{x}{\tau})$ where τ is pulse width.

Bandwidths: → if $g_1(t)$ and $g_2(t)$ have bandwidths B_1 and B_2, then $g_1(t)\cdot g_2(t)$ has bandwidth $B_1 + B_2$.

→ width of $x(t) * y(t)$ is sum of widths of $x(t)$ and $y(t)$

→ if bandwidth of $g(t)$ is B Hz, then bandwidth of $g^n(t)$ is nB Hz.

• $|Y(f)| = |X(f)|\cdot|H(f)|$ and $\theta_y(f) = \theta_x(f) + \theta_h(f)$

Distortionless Transmission: $y(t) = kx(t - t_d)$ → identical in shape w/ delay; and $Y(f) = kX(f)e^{-j2\pi f t_d}$

Parseval's Theorem (frequency domain): $E_g = \int_{-\infty}^{\infty} |G(f)|^2 df$; if through a filter: $Y(f) = G(f)H(f)$, $E_y = 2|G(f_0)|^2 \Delta f$ (if $H(f) = 1$), where $|G(f_0)|^2$ is the ESD and $2|G(f_0)|^2$ b/c of + and -. Otherwise, $E_y = \int_{-\infty}^{\infty} |G(f)H(f)|^2 df$

ENERGY SPECTRAL DENSITY: $\psi_g(\tau) = |G(f)|^2$; $E_g = \int_{-\infty}^{\infty} \psi_g(f)\, df$

Wiener-Khintchine Theorem: Autocorrelation of $g(t)$ and its ESD $\Psi_g(f)$ are Fourier Transform paie: $\psi_g(\tau) \leftrightarrow \Psi_g(f) = |G(f)|^2$; ** $\psi_g(\tau) = g(\tau) * g(-\tau)$

Power: $P_g = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|G(f)|^2}{T}\, df$; **POWER SPECTRAL DENSITY:** $S_g(f) = \lim_{T\to\infty} \frac{|G(f)|^2}{T}$, so $P_g = \int_{-\infty}^{\infty} S_g(f)\, df = 2\int_0^{\infty} S_g(f)\ df$ if it is an even function

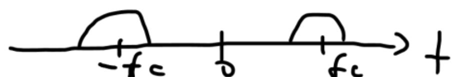
Time Autocorrelation of Power Signals: $R_g(-\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t) \cdot g(t-\tau)\, dt$; $R_g(-\tau) = R_g(\tau)$;

→ PSD of $S_g(f)$ is Fourier Transform of $R_g(\tau)$; $R_g(\tau) = \lim_{T\to\infty} \frac{\Psi_{gT}(\tau)}{T}$, → From Wiener-Khintchine: $R_g(\tau) \leftrightarrow \lim_{T\to\infty} \frac{|G(f)|^2}{T} = S_g(f)$

→ PSD of Modulated Signal: if $\psi(t) = g(t)\cos(2\pi f_0 t)$ and if $f_0 \geq B$, then PSD $S_\psi(f) = \frac{1}{A}[S_g(f+f_0) + S_g(f-f_0)]$

Modulation: $s(t) = m(t)\cos(2\pi f_c t)$, where $m(t)$ is modulating/message signal, $\cos(2\pi f_c t)$ is carrier, and $s(t)$ is modulated signal.

DSB-SC: $\psi_{DSB-SC}(t) = A_c m(t)\cos(2\pi f_c t)$;



AM: $\left(A + m(t)\right) cos(2\pi f_c t) \leftrightarrow \frac{1}{2}[M(f+f_c) + M(f-f_c)] + \frac{A}{2}[\delta(f+f_c) + \delta(f-f_c)]$: → For signal $E(t)\cos\omega_c t$, "envelope" is $|E(t)|$ if $E(t)$ varies slowly

→ Envelope Detection: 1) $f_c \gg$ Bandwidth of $m(t)$ ~ condition for envelope detection, 2) $A + m(t) \geq 0$

→ **Power:** $\frac{A^2}{2} + \frac{m^2(t)}{2}$ if message is orthogonal with carrier, where $P_c = \frac{A^2}{2}$ (carrier power) and $P_s = \frac{m^2(t)}{2}$ (sideband power); Power efficiency: $\frac{P_s}{P_c+P_s}$; for tone modulation $m(t) = \mu A\cos\omega_m t$: $\eta = \frac{\mu^2}{2+\mu^2}$; Modulation Index: $\mu = \frac{m_p}{A}$ $(0 \leq \mu \leq 1)$ if distortionless demodulation ; if nonzero offset, then : $\mu = \frac{m_{max} - m_{min}}{2A + m_{max} + m_{min}}$

Hilbert Transform: $x_h(t) = \mathcal{H}[x(t)] = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\alpha)}{t-\alpha}\ d\alpha = x(t) * \frac{1}{\pi t}$; **$\frac{1}{\pi t} \leftrightarrow -j\text{sgn}(f) = \begin{cases} j, & f > 0 \\ -j, & f < 0 \end{cases}$

Quadrature Amplitude Modulation (QAM): $\psi_{QAM}(t) = m_1(t)\cos(\omega_c t) + m_2(t)\sin(\omega_c t)$ → 2 baseband signals can be separated at the receiver if 2 local carriers are used in phase quadrature: $x_1(t) = 2\psi_{QAM}(t)\cos(\omega_c t)$ and $x_2(t) = 2\psi_{QAM}(t)\sin(\omega_c t)$

Vestigial Sideband (VSB) Amplitude Modulation: $m(t) \rightarrow \underset{2\cos\omega_c t}{\otimes} \rightarrow \boxed{H_i(t)} \rightarrow \psi_{VSB}(t)$

$\Phi_{VSB}(f) = [M(f+f_c) + M(f-f_c)]H_i(f)$; Demodulation: $\Phi_{VSB}(f) = \frac{M(f+f_c)+M(f-f_c)}{2} + \frac{M_v(f+f_c)+M_v(f-f_c)}{2j} \leftrightarrow m(t)\cos(2\pi f_c t) + m_v(t)\sin(2\pi f_c t)$, where $m_v(t)$ is a lowpass signal

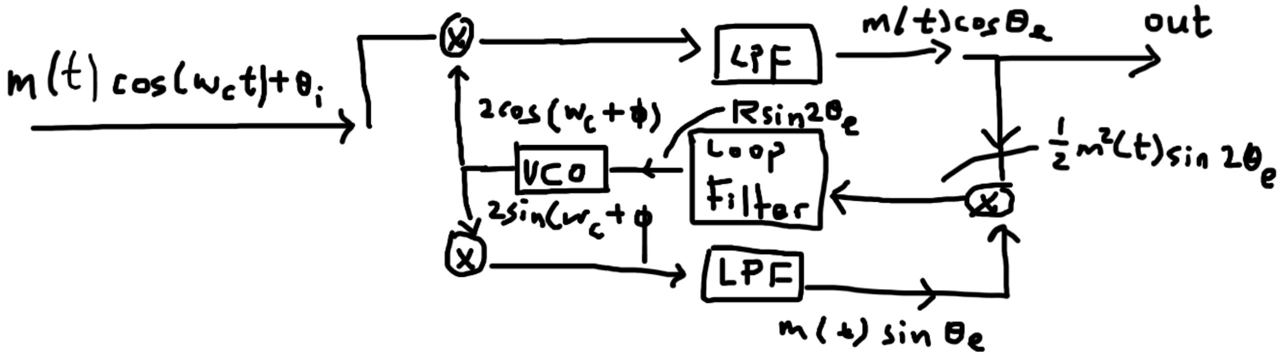
Instantaneous Frequency: $\omega_i(t) = \frac{d\theta}{dt}, \theta(t) = \int_{-\infty}^{t} \omega_i(\alpha)\, d\alpha$

PM: $\psi_{PM} = A\,cos(\omega_c t + k_P m(t))$; $\Delta f = k_p \frac{m(t)_{max} - m(t)_{min}}{2\cdot 2\pi}$, $\beta = \frac{\Delta f}{B}$, $B_{PM} = 2(\Delta f + B) = 2B(\beta + 1)$; Power = $\frac{A^2}{2}$

FM: $\psi_{FM} = A\cos(\omega_C t + k_f \int_{-\infty}^{t} m(\alpha)\, d\alpha)$; $\Delta f = k_f \frac{m_{max} - m_{min}}{2\cdot 2\pi}$, $B_{FM} \cong 2(\Delta f + 2B)$ (for peak frequency deviation); $B_{FM} = 2(\Delta f + B) = 2(\frac{k_f m_p}{2\pi} + B)$ $\beta = \frac{\Delta f}{B}$

PLL

Diagram:



Sampling:

`Original: $g(t)$; Sampled: $\bar{g}(t) = g(t) \cdot \delta_{T_s}(t)$; $\bar{G}(f) = \mathcal{F}\{\bar{g}(t)\} = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} G(f - nf_s) = \frac{1}{T_s} G(f - \frac{n}{T_s})$

→For $f_s \geq 2B$, can recover with Brick Wall Filter: $H(f) = T_s \Pi\left(\frac{f}{2B}\right) \rightarrow H(f) \cdot \bar{G}(f) = G(f)$, $h(t) = sinc(2\pi Bt)$

$$\rightarrow g(t) = \sum_{n}^{\infty} g(nT_s) sinc(2\pi B(t - nT_s))$$

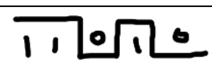
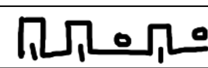
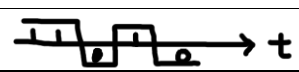
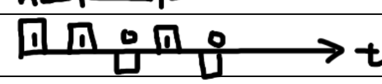
Equalizer: undo distortion and remove copies of LPF: $\mathrm{E}(f) \cdot \widetilde{G(f)} = G(f)e^{-j2\pi f t_0} = E(f)P(f)\frac{1}{T_s}\sum_n G(f - nf_s)$;

$$E(f) = \begin{cases} T_s \dfrac{e^{-j2\pi f t_0}}{P(f)}, |f| < B \\ flexible, B \leq |f| \leq f_s - B \\ 0, |f| \geq f_s - B \end{cases}$$

MODULATIONS: Pulse Amplitude Modulation (pulse amplitude is signal amplitude); Pule Width Modulation (same location, different widths); Pulse Position Modulation (same widths, different positions); Pulse Code Modulation (analog signal quantized over a range) … mapped to a bit sequence; bits sent as pulses (on-off signaling) where 1-pulse and 0-no pulse

Bandwidth of PCM: → $m(t)$ bandwidth of B; need to be ≥ $2B$ samples/s to reconstruct; →each sample quantized to n bits; → $bit\ rate \geq 2nB$ bits/s; max info rate needs nB Hz channel BW; →need n times as much BW for PCM as original signal.

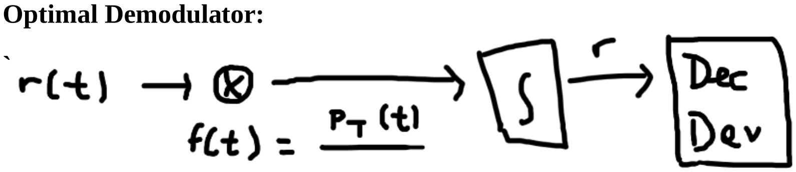
Line coding: **Symbol — "piece of info", digital or analog. Line Coding – class of digital modulation for baseband channels (usually binary): 0 → $s_0(t)$ and 1 → $s_1(t)$; has 2 broad classes:

Polarity/Type (for $u = [1\ 1\ 0\ 1\ 0]$)	**Non-Return to Zero (NRZ)**	**Return-to-Zero**
Unipolar		
Bipolar		

PSD of Line Coding: If the transmitted signal is $y(t) = \sum_k a_k p(t - kT_b)$ where T_b is bit duration and a_k is line code amplitude, then $y(t) = p(t) * \sum_k a_k \delta(t - kT_b) = p(t) * x(t)$ and $S_y(f) = |p(f)|^2 S_x(f)$ and $S_x(f) \leftrightarrow R_x(\tau)$

Optimal Demodulator:

`

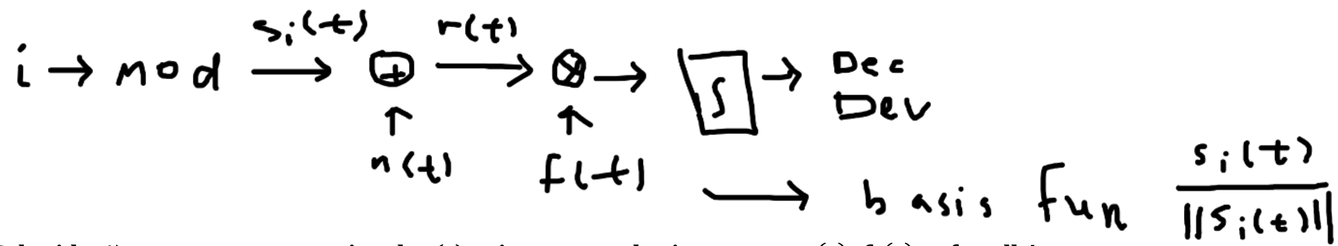


BPSK: $s_0(t) = \sqrt{\frac{2E}{T}} cos(\omega_c t) p_T(t)$ and $s_1(t) = \sqrt{\frac{2E}{T}} cos(\omega_c t + \pi) p_T(t) = -\sqrt{\frac{2E}{T}} cos(\omega_c t) p_T(t)$, Antipodal since $s_0(t) = -s_1(t)$;

$E = ||s_0||^2 = ||s_1||^2 = \frac{1}{2}||\sqrt{\frac{2E}{T}} p_T(t)||^2 = \frac{E}{T}\int_0^T 1^2 dt = E$, so $s_0 = \sqrt{E}$ and $s_1 = -\sqrt{E}$ (same as Bipolar NRZ)

Line-coded: $S_y(f) = \frac{|P(f)|^2}{T_b}(R_0 + 2\sum_{n=1}^{\infty} \mathrm{R_n} \cos(2\pi \mathrm{nf} \mathrm{T_b}))$ **Polar Signaling: $S_y(f) = \frac{|P(f)|^2}{T_b}$ On-Off: $S_y(f) = \frac{|P(f)|^2}{4T_b}\left(1 + \frac{1}{T_b}\sum_{n=-\infty}^{\infty} \delta\left(\mathrm{f} - \frac{\mathrm{n}}{\mathrm{T_b}}\right)\right)$ Bipolar signaling: $S_y(f) = \frac{|P(f)|^2}{4T_b}\left(\sin^2(\pi \mathrm{f} \mathrm{T_b})\right)$**

For Binary Modulation: ML Decision Rule (ONLY USE THIS) … for AWGN channel, this is minimum distance rule.



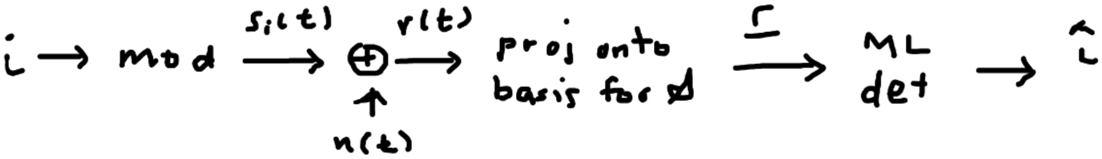
Gram-Schmidt: 1) represent current signal $s_i(t)$ using current basis: $s_{ij} = < s_i(t), f_j(t) >$ for all j.

2) Find error signal: $e_i(t) = s_i(t) - \widehat{s_i(t)}$

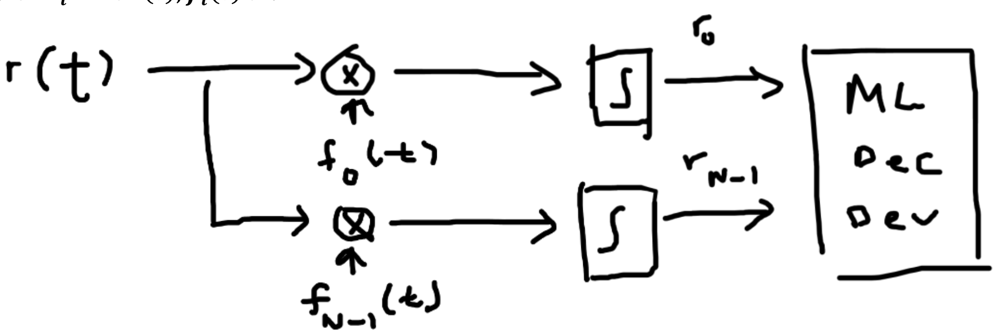
3) If $||e_i(t)||^2 = 0$, go to the next signal, else create a new basis function, $e_i(t)$ must be orthogonal to all existing $f_j(t)$. Let $f_k(t) = \frac{e_i(t)}{||e_i(t)||}$, then $f_k(t)$ orthogonal to $f_j(t)$ for all j and $||f_k(t)||^2 = 1$ … new set of $\{f_j(t), \dots, f_k(t)\}$, can represent $s_i(t)$ with 0 error energy, then $s_i = [s_{i0}, s_{i1}, \dots, s_{iN-1}]$ ~ if did not find one, put 0 in its place. At each iteration either make a new $f_k(t)$ or don't make it.

**** Dimensionality of a signal set $\mathcal{S}$ is the cardinality of a basis for $\mathcal{S}$. $N = dim\, \mathcal{S} = |\mathcal{F}|$**

Digital Receiver Design



With $r_i = < r(t), f_i(t) >$:



Energy Efficiency: probability of error as a function of signal-to-noise ratio; Spectral Efficiency: frequency/BW

Signal set with M signals is M-ary modulation; and $m = log_2(M)$ bits/symbol.

Bit Rate/data rate $R_b = m \frac{bit}{sym} * R_s \frac{sym}{s}$ and $\eta = \frac{R_b \frac{bits}{s}}{B\, Hz}$ (this is spectral efficiency, B = bandwidth)

Decision rule: MLE $P_{error} = Q\left(\frac{d/2}{\sqrt{N/2}}\right)$, related to distance on signal constellation too. **If given dB values, compute in linear units. Report $\frac{E_b}{N_0}$ needs to be in dB units. **Q( ) function is the survival function or 1-normalcdf() if using calculator. **For binary modulation, $P_s = Q\left(\frac{\frac{d}{2}}{\sqrt{\frac{N_0}{2}}}\right) = Q\left(\frac{\sqrt{\frac{E}{2}}}{\sqrt{\frac{N_0}{2}}}\right) = Q(\sqrt{\frac{E}{N_0}})$

OTHERS: $||s_i(t)||^2 = \int_0^T (s_i(t))^2 dt$ and $E_S = E_b log_2(M)$

Energy efficiency: average neighbors Q function or distance. Spectral efficiency: related to η, max around $\boldsymbol{log_2(M)}$

OFDM: assign data at different frequencies channels, IFFT, add guard time, circular convolution.