DSP notes 2

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1 Convolution Intuition: The Algebraic Perspective

1.1 Core Insight: Collecting All System Interactions

Convolution fundamentally captures **all possible ways two systems can interact through time**. Rather than viewing it as a complex mathematical operation, we can understand it through the lens of algebraic coefficient collection.

1.2 The Polynomial Multiplication Analogy

1.2.1 Discrete Case

Consider two polynomials representing discrete signals: x[n] $\leftrightarrow X(z) = \sum_{i=0}^{N-1} x[i]z^i$ $h[n] \leftrightarrow H(z) = \sum_{j=0}^{M-1} h[j]z^j$

When we multiply these polynomials:

$$Y(z) = X(z) \cdot H(z) = \left(\sum_{i=0}^{N-1} x[i]z^i\right)\left(\sum_{j=0}^{M-1} h[j]z^j\right) \tag{1}$$

Each term $x[i] \cdot h[j]$ contributes to the coefficient of z^{i+j}. The coefficient of z^n becomes:

$$y[n] = \sum_{i+j=n\, i,j\geq 0} x[i]h[j] = \sum_{i=0}^{n} x[i]h[n-i] \tag{2}$$

This is exactly the convolution formula!

1.3 The "All Possible Combinations" Principle

1.3.1 Discrete Systems

In discrete signal processing, convolution systematically accounts for every possible interaction:

- Sample $x[0]$ affects outputs at times $0, 1, 2, \ldots$ with weights $h[0], h[1], h[2], \ldots$
- Sample $x[1]$ affects outputs at times $1, 2, 3, \ldots$ with weights $h[0], h[1], h[2], \ldots$
- Sample $x[k]$ affects outputs at times $k, k+1, k+2, \ldots$ with weights $h[0], h[1], h[2], \ldots$

At output time n, we collect **all** contributions:

$$y[n] = \sum_{k=0}^{n} x[k] \cdot h[n-k] \tag{3}$$

1.3.2 Time as the Collecting Index

The key insight is that **time serves as the coefficient index**:

- Each interaction $x[i] \times h[j]$ contributes to output time $(i + j)$
- We collect all interactions that sum to the same output time
- This gives us the convolution sum at each time index

1.4 Extension to Continuous Systems

1.4.1 Infinitesimal Interactions

For continuous systems, we apply the **same principle** but with infinitesimal elements:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{4}$$

1.4.2 Conceptual Parallel

Discrete	**Continuous**
Finite samples $x[i]$	Infinitesimal elements $x(\tau)d\tau$
Discrete weights $h[j]$	Continuous weights $h(t-\tau)$
Sum over indices	Integrate over time
$\sum x[i]h[n-i]$	$\int x(\tau)h(t-\tau)d\tau$

At any continuous output time t, we collect contributions from:

- $x(0)$ weighted by $h(t-0) \times dt$
- $x(dt)$ weighted by $h(t-dt) \times dt$
- $x(\tau)$ weighted by $h(t-\tau) \times d\tau$ for all τ

1.5 Fundamental Principles

1.5.1 Universal Interaction Accounting

Whether discrete or continuous, convolution provides **exhaustive accounting** of system interactions:

1. **Completeness:** Every possible interaction is included
2. **Organization:** Interactions are organized by output time
3. **Linearity:** All interactions are summed/integrated linearly

1.5.2 Why Convolution Appears Everywhere

Convolution emerges naturally whenever:

- Two systems interact over time
- Past inputs influence future outputs
- Linear superposition applies

Examples: signal filtering, probability distributions, heat diffusion, acoustic reverberation.

1.6 Key Takeaway

Convolution is the mathematical description of how two systems can interact completely through time. It systematically accounts for every possible cross-interaction, organized by the time at which each interaction's effect appears in the output.

In essence: *We sum all possible combinations of two systems, where time serves as the organizing coefficient.*